Riemannian Geometry $\boxed{\text{Week 4}}$

Flow of a vector field

Let $X \in \Gamma(M)$. <u>Recall</u>: $\gamma : (a,b) \to M$ is an integral curve of X if $\dot{\gamma}(t) = X|_{\gamma(t)}$ $\forall t \in (a,b)$.

From the theory of ODEs:

<u>Local well-posedness thm</u>: Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be smooth.

Then, $\forall u_0 \in \mathbb{R}^n$, the initial value problem

$$\begin{cases} \dfrac{du}{dt} = F(t, u(t)) \\ u(0) = u_0 \end{cases}$$

has a <u>unique</u> <u>maximal</u> solution $u : (-T_-, T_+) \to \mathbb{R}^n$, $T_-, T_+ \in (0, +\infty]$,

If $T_+ < +\infty$, then $\limsup_{t \to T_+} \|u(t)\| = +\infty$ (similarly for T_-).

Moreover, $\forall$ interval $[a,b] \subsetneq (-T_-, T_+)$, $u|_{[a,b]}$ depends <u>smoothly</u> on u_0.

- That is to say: As long as the solution remains in a compact set, it can be extended.
- $u(t) = u(t; u_0)$: smooth in its arguments.

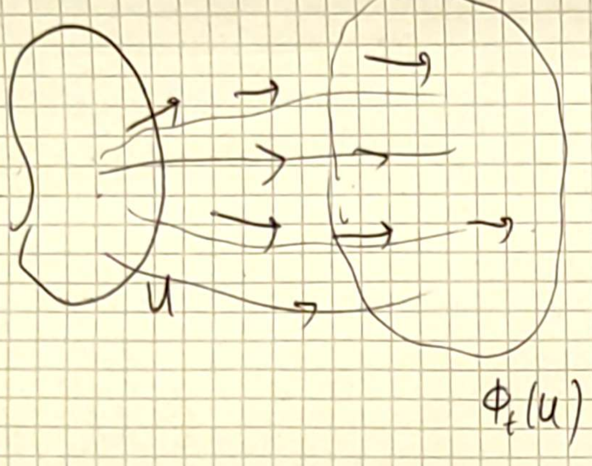
By working in local coordinates, the above theorem implies the following:

<u>Thm</u>: Let $X \in \Gamma(M)$. $\forall\, U \subseteq M$ open with $\bar{U}$ compact,

$\exists\, \delta > 0$ and $\exists!$ smooth map $\Phi : (-\delta, \delta) \times U \to M$ (<u>Flow map</u>)

such that $\forall q \in U$, $\gamma(t) = \Phi_t(q)$ is an integral curve of X starting from q, i.e.

$$\begin{cases} \dfrac{d}{dt} \Phi_t(q) = X|_{\Phi_t(q)} \\ \Phi_0(q) = q \end{cases} \quad \left(\leftrightarrow \frac{d\Phi_t^k(q)}{dt} = X^k(\Phi_t(q)). \right)$$



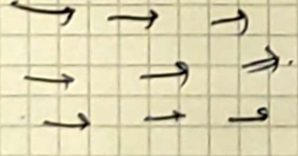
Moreover: If X is compactly supported: $U = M$, $\delta = +\infty$.

Remark: Two integral curves cannot intersect (uniqueness of solutions of ODEs) $\Rightarrow$ Φ_t is a diffeomorphism on its image

Example: If $X = \frac{\partial}{\partial x^1}$

$$\frac{\partial}{\partial t} \Phi_t^\mu(x) = \left(\frac{\partial}{\partial x^1}\right)^\mu \Big|_{\Phi_t(x)} = (1, 0, \dots, 0) = \delta_1^\mu$$

So $\Phi_t((x^1, \dots, x^n)) = (x^1 + t, x^2, \dots, x^n)$



Locally: If $X \neq 0$, there exists a coordinate system where $X = \frac{\partial}{\partial x^1}$

If we drop compactness, the flow doesn't necessarily exist for all times:

Ex: On $\mathbb{R}$: $X = y^2 \frac{d}{dy}$

Integral curve $y = y(t)$: $\dot{y} = y^2 \xRightarrow{\text{if } y(0) = y_0 > 0} y(t) = \dfrac{1}{\frac{1}{y_0} - t}$.

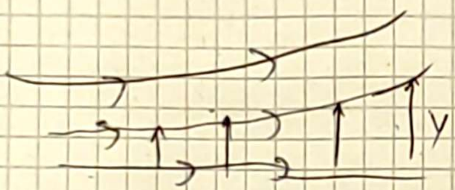
So vector fields: Act as generators of diffeomorphisms!

In local coordinates: $[X,Y]^k = \lim_{t,s\to 0} \dfrac{(\Phi^{(Y)}_{-s}\phi^{(X)}_{-t}\phi^{(Y)}_{s}\phi^{(X)}_{t}(p))^k - p^k}{ts}$

Lie derivative: Derivative along the flow of $X \in \Gamma(M)$

Def • For $f \in C^\infty(M)$: $\mathcal{L}_X f(p) = \lim_{t\to 0} \dfrac{f(\phi_t(p)) - f(p)}{t}$

• For $Y \in \Gamma(M)$: $\mathcal{L}_X Y = \lim_{t\to 0} \dfrac{d\phi_{-t}(Y|_{\phi_t(p)}) - Y|_p}{t}$



$\leftarrow \mathcal{L}_X Y = 0$

Theorem: • For $f \in C^\infty(M)$: $\mathcal{L}_X f = X(f)$

• For $Y \in \Gamma(M)$: $\mathcal{L}_X Y = [X,Y]$

Proof: We will consider cases:

• If $X|_p \neq 0$: Fix local coordinates around p so that $X = \dfrac{\partial}{\partial x^1}$ on a neighborhood U of p.

(**Remark:** It suffices to verify the identity in one coordinate system: Since it is a geometric identity, it will then be true in all coordinate systems)

Then $\Phi^{(X)}_t((x^1, \dots, x^n)) = (x^1 + t, x^2, \dots, x^n)$

So $\mathcal{L}_X f = \lim_{t \to 0} \dfrac{f(x^1+t, x^2, \dots, x^n) - f(x^1, x^2, \dots, x^n)}{t} = \dfrac{\partial}{\partial x^1} f = X(f)$

Also: $(d\varphi_{-t})^\beta_\alpha = \dfrac{\partial \varphi^\beta_{-t}}{\partial x^\alpha} = \delta^\beta_\alpha$.

So $(\mathcal{L}_X Y)^k = \lim_{t \to 0} \dfrac{(d\varphi_{-t}(Y|_{\varphi_t(p)}))^k - Y^k|_p}{t} = \lim_{t \to 0} \dfrac{Y^k(x^1+t, x^2, \dots, x^n) - Y^k(x^1, \dots)}{t}$

$$= \frac{\partial Y^k}{\partial x^1} = \left[\frac{\partial}{\partial x^1}, Y\right]^k.$$

So the statement is true on $\operatorname{supp} X$.

$\Longrightarrow$ By continuity: True on $\overline{\operatorname{supp} X}$.

For any $p \in M \setminus \overline{\operatorname{supp} X}$: $\exists$ open neighborhood of p such that $X \equiv 0$ there $\Rightarrow$ $\varphi_t = \mathrm{Id}$ there

$\Rightarrow \mathcal{L}_X f, \mathcal{L}_X Y = 0$ ∎

We can extend the definition of the Lie derivative to <u>any</u> tensor field (of any type) by the requirements:

- On 1-forms: $(\mathcal{L}_X \omega)(Y) = \mathcal{L}_X(\omega(Y)) - \omega(\mathcal{L}_X Y)$ ①
- $\mathcal{L}_X(f \otimes g) = (\mathcal{L}_X f) \otimes g + f \otimes (\mathcal{L}_X g)$

Inductively: It extends on $\otimes^k TM \otimes^l T^*M$

And it respects contractions:

$$\mathcal{L}_X(\operatorname{tr} f) = \operatorname{tr}(\mathcal{L}_X f)$$

Note: ① is just the statement that the (1,1) contraction is respected.

Because: If $\omega \in \Gamma^*(M)$, $Y \in \Gamma(M)$:

$$(\omega \otimes Y)^{\alpha}_{\beta} = \omega_\beta Y^\alpha$$

$$\operatorname{tr}(\omega \otimes Y) = (\omega \otimes Y)^{\alpha}_{\alpha} = \omega_\alpha Y^\alpha = \omega(Y)$$

(In general: Expressions like $\omega(Y)$, $g(X,Y)$ can be viewed as contractions of products

E.g. $g(X,Y) = g_{\alpha\beta} X^\alpha Y^\beta = \operatorname{tr}(\operatorname{tr}(g \otimes X \otimes Y))$

So $\mathcal{L}_X(g(Y,Z)) = (\mathcal{L}_X g)(Y,Z) + g(\mathcal{L}_X Y, Z) + g(Y, \mathcal{L}_X Z)$)

Note:

If $f \in \otimes^k TM \otimes^l T^*M$, $X = \frac{\partial}{\partial x^1}$ in local coordinates

then $(\mathcal{L}_X f)^{i_1 \ldots i_k}_{j_1 \ldots j_l} = \frac{\partial f^{i_1 \ldots i_k}_{j_1 \ldots j_l}}{\partial x^1}$

Def: Let (M,g) be a Riem. manifold. $X \in \Gamma(M)$ is a **Killing** vector field if $\mathcal{L}_X g = 0$

(generator of isometries)

Connections

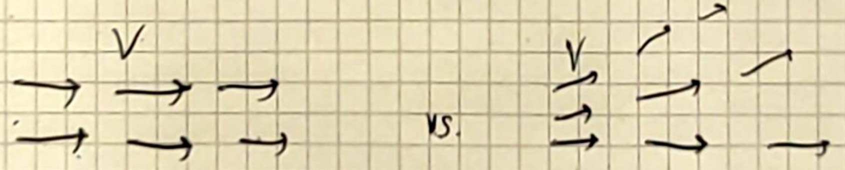
We will need to "compare" tangent vectors at nearby points.

Example: Compute the acceleration of a curve, define parallel transport etc.

In particular, we need to define an appropriate notion of a directional derivative of a vector field.

$D_V X|_p$: Should only depend on $V|_p$ (the direction)

$\mathcal{L}_V X$: Not a good candidate $\quad (\mathcal{L}_{f \cdot V} X \neq f \cdot \mathcal{L}_V X)$



Definition: A connection ∇ on M is a map $\nabla : \Gamma(M) \times \Gamma(M) \longrightarrow \Gamma(M)$ such that:

1) ∇ is $C^\infty(M)$ - linear in the first argument.

$$\nabla_{X_1 + f \cdot X_2} Y = \nabla_{X_1} Y + f \cdot \nabla_{X_2} Y$$

(So $\forall Y \in \Gamma(M)$: ∇Y is a $(1,1)$-tensor field)

2) ∇ is $\mathbb{R}$-linear in the second argument.

$$\nabla_X (Y_1 + a Y_2) = \nabla_X Y_1 + a \cdot \nabla_X Y_2 \quad , a \in \mathbb{R}.$$

3) Satisfies the Leibniz rule with respect to 2^{nd} argument:

$$\nabla_X (f \cdot Y) = X(f) \cdot Y + f \cdot \nabla_X Y.$$

So it's a derivative.

<u>Lemma</u>: If ∇ is a connection on M: It defines a unique connection on any open subset $U \subseteq M$.

<u>Proof</u>: Let $X, Y \in \Gamma(U)$, $p \in U$. I want to say that $\nabla_X Y|_p$ is uniquely defined.

Let $f : M \longrightarrow [0,1]$ be $\begin{cases} \equiv 1 & \text{on a neighborhood of } p \text{ in } U \\ \equiv 0 & \text{on } M \setminus U. \end{cases}$

Then $\tilde{X} = f \cdot X$, $\tilde{Y} = f \cdot Y$ extend on the whole of M

Define $\nabla_X Y|_p = \nabla_{\tilde{X}} \tilde{Y}|_p$.

Independent of the choice of f:

If $Y, \tilde{Y}$ two vector fields such that

$$\tilde{Y} = Y \text{ in a neighborhood } V \text{ of } p,$$

then if $f : M \to [0,1]$, $f \begin{cases} \equiv 1 & \text{on } V' \subset\subset V \\ \equiv 0 & \text{on } M \setminus V \end{cases}$,

then
$$\nabla_X \left(f \cdot (Y - \tilde{Y}) \right)\Big|_p = X(f)\big|_p \cdot \underbrace{(Y - \tilde{Y})\big|_p}_{0} + \underbrace{f\big|_p}_{1} \left(\nabla_X (Y - \tilde{Y}) \right)\Big|_p$$

But $f \cdot Y - \tilde{Y} \equiv 0 \implies \nabla_X Y\big|_p = \nabla_X \tilde{Y}\big|_p$ $\blacksquare$

We will use the above in order to talk about $\nabla_X Y$ when X, Y are not defined on the whole of M (e.g. coordinate vector fields)

Definition: Christoffel symbols

Let $(x^1, \dots, x^n)$ be a local coordinate system.

Define Γ^k_{ij} by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \qquad \left(\Leftrightarrow \quad \Gamma^k_{ij} = dx^k \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) \right)$$

Γ^k_{ij}: Not components of a tensor (see Exercises)

(But: $\nabla - \bar{\nabla}$ is a tensor, so $\Gamma^k_{ij} - \bar{\Gamma}^k_{ij}$ is also a tensor)

In local coordinates:

$$\boxed{(\nabla_X Y)^k = X^i \partial_i Y^k + \Gamma^k_{ij} X^i Y^j}$$

"Zeroth order correction"

<u>Example</u>:

Flat connection on $\mathbb{R}^2$: $(\nabla_X Y) = X^i \partial_i Y$ in Cartesian coordinate

- So in Cartesian coordinates: The Christoffel symbols vanish
- In polar coordinates: <u>Not</u>.

<u>Definition</u>: Torsion of a connection:

$$T(X,Y) = \nabla_X Y - \underbrace{\nabla_Y X - [X,Y]}_{}$$

$\nabla^*_X Y$: "Dual" connection (Check that it is indeed a connection)

Note: T is an <u>anti-symmetric</u> $(1,2)$ tensor!

(Exercise).

Torsion-free (or symmetric) connection: $T = 0$

<u>Note</u>: $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$ for coordinate vector fields

(necessary and sufficient condition for two vector fields to be coordinate ones)

So $\boxed{T = 0 \iff \Gamma^k_{ij} = \Gamma^k_{ji}}$ (It suffices to be true in one coordinate system)